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Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process

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The authors propose that conceptual and procedural knowledge develop in an iterative fashion and that improved problem representation is 1 mechanism underlying the relations between them. Two experiments were conducted with 5th- and 6th-grade students learning about decimal fractions. In Experiment 1, children's initial conceptual knowledge predicted gains in procedural knowledge, and gains in procedural knowledge predicted improvements in conceptual knowledge. Correct problem representations mediated the relation between initial conceptual knowledge and improved procedural knowledge. In Experiment 2, amount of support for correct problem representation was experimentally manipulated, and the manipulations led to gains in procedural knowledge. Thus, conceptual and procedural knowledge develop iteratively, and improved problem representation is 1 mechanism in this process.

Understanding the process of knowledge change is a central goal in the study of development and education. Two essential types of knowledge that children acquire are *conceptual understanding* and *procedural skill.* Competence in domains such as mathematics rests on children developing and linking their knowledge of concepts and procedures (Silver, 1986). However, competing theories have been proposed regarding the developmental relations between conceptual and procedural knowledge.

The majority of past research and theory on these relations has focused on whether conceptual or procedural knowledge emerges first (Rittle-Johnson & Siegler, 1998). The developmental precedence of one type of knowledge over another has been hotly debated (e.g., Gelman & Williams, 1998; Siegler, 1991; Siegler & Crowley, 1994; Sophian, 1997). In contrast to this past research and theory, we propose that throughout development, conceptual and procedural knowledge influence one another. Specifically, we propose that conceptual and procedural knowledge develop iteratively, with increases in one type of knowledge leading to increases in the other type of knowledge, which trigger new increases in the first (see Figure 1). This iterative model highlights the need to identify mechanisms that underlie knowledge change. In this research we examined the role of change in problem representation as one potential change mechanism.

In this study we evaluated the iterative model in two experiments on children's learning about decimal fractions. Experiment 1 provides correlational evidence for the relations proposed within the iterative model. Experiment 2 provides causal evidence for one link in the model: the link from improved problem representation to improved procedural knowledge.

Relations Between Conceptual and Procedural Knowledge

Many theories of learning and development distinguish between conceptual and procedural knowledge (e.g., Anderson, 1993; Bisanz & LeFevre, 1992; Greeno, Riley, & Gelman, 1984; Karmiloff-Smith, 1994; Piaget, 1978). These two types of knowledge lie on a continuum and cannot always be separated; however, the two ends of the continuum represent distinct types of knowledge. We define *procedural knowledge* as the ability to execute action sequences to solve problems. This type of knowledge is tied to specific problem types and therefore is not widely generalizable. To assess procedural knowledge researchers typically use routine tasks, such as counting a row of objects or solving standard arithmetic computations, because children are likely to use previously learned step-by-step solution methods to solve the problems (e.g., Briars & Siegler, 1984; Hiebert & Wearne, 1996). In contrast to procedural knowledge, we define *conceptual knowledge* as implicit or explicit understanding of the principles that govern a domain and of the interrelations between units of knowledge in a

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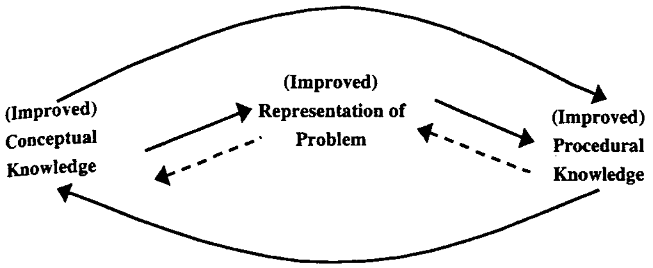

Figure 1. Iterative model for the development of conceptual and procedural knowledge. The solid links are examined in this study.

domain. This knowledge is flexible and not tied to specific problem types and is therefore generalizable. Furthermore, it may or may not be verbalizable. To assess conceptual knowledge, researchers often use novel tasks, such as counting in nonstandard ways or evaluating unfamiliar procedures. Because children do not already know a procedure for solving the task, they must rely on their knowledge of relevant concepts to generate methods for solving the problems (e.g., Bisanz & LeFevre, 1992; Briars & Siegler, 1984; Gelman & Meck, 1983; Greeno et al., 1984; Hiebert & Wearne, 1996; Siegler & Crowley, 1994).

Most theories of the development of conceptual and procedural knowledge have focused on which type of knowledge develops first in a given domain. According to *concepts-first* theories, children initially develop (or are born with) conceptual knowledge in a domain and then use this conceptual knowledge to generate and select procedures for solving problems in that domain (e.g., Geary, 1994; Gelman & Williams, 1998; Halford, 1993). Evidence consistent with the developmental precedence of conceptual knowledge has been found in mathematical domains ranging from simple arithmetic to proportional reasoning (e.g., Byrnes, 1992; Cowan & Renton, 1996; Dixon & Moore, 1996; Hiebert & Wearne, 1996; Siegler & Crowley, 1994; Wynn, 1992). This theory and evidence has been used to justify reforms in mathematics education that focus on inculcating conceptual knowledge before teaching procedural knowledge (National Council of Teachers of Mathematics [NCTM], 1989; Putnam, Heaton, Prewat, & Remillard, 1992).

Alternatively, conceptual knowledge may develop after procedural knowledge. According to *procedures-first* theories, children first learn procedures for solving problems in a domain and later extract domain concepts from repeated experience solving the problems (e.g., Fuson, 1988; Karmiloff-Smith, 1992; Siegler & Stern, 1998). Evidence consistent with a given procedure preceding knowledge of key concepts underlying that procedure has been found in a variety of mathematical domains such as counting and fraction multiplication (e.g., Briars & Siegler, 1984; Byrnes & Wasik, 1991; Frye, Braisby, Love, Maroudas, & Nicholls, 1989; Fuson, 1988; Hiebert & Wearne, 1996).

How can these opposing theories and bodies of evidence be reconciled? In domains such as counting and simple arithmetic, discussions of these contradictory findings have focused on methodological limitations of research leading to opposing conclusions (e.g., Gelman & Meck, 1986; Siegler, 1991). However, a more basic problem may involve the difficulty of defining what it means to "have" or "not have" a particular type of knowledge (see Sophian, 1997).

This debate over which type of knowledge develops first may obscure the gradual development of each type of knowledge and the interactions between the two knowledge types during development. The iterative model shown in Figure 1 indicates how such a process may occur. Increases in one type of knowledge lead to gains in the other type of knowledge, which in turn lead to further increases in the first. Knowledge of a particular type is often incomplete, and a variety of experiences, such as problem solving, observation of other people's activities, direct verbal instruction, and reflection, may initiate knowledge change.

Past research is consistent with this gradual, bidirectional model of conceptual and procedural knowledge development. First, children often have partial knowledge of both concepts and procedures (e.g., Fuson, 1990; Gelman & Gallistel, 1978). Second, greater knowledge of one type is associated with greater knowledge of the other (Baroody & Gannon, 1984; Byrnes & Wasik, 1991; Cauley, 1988; Cowan & Renton, 1996; Cowan, Dowker, Christakis, & Bailey, 1996; Dixon & Moore, 1996; Hiebert & Wearne, 1996). Third, improving children's knowledge of one type can lead to improvements in the other type of knowledge (Rittle-Johnson & Alibali, 1999). Conceptual and procedural knowledge may develop in a hand-over-hand process, rather than one type strictly preceding the other.

The iterative model also helps to resolve two issues. First, early knowledge tends to be very limited, so the fact that children know something about *X* does not mean that they fully understand *X*. The early knowledge is real, but partial. Thus, at a particular point in time, one type of knowledge might be better developed than the other, but it is not meaningful to say children "have" one type of knowledge but "do not have" the other type.

Second, either conceptual or procedural knowledge may begin to develop first. This view eliminates fruitless arguments about whether conceptual or procedural knowledge generally precedes the other. The relative timing and frequency of exposure to concepts and procedures in a domain determines whether initial knowledge is conceptual or procedural in nature (Rittle-Johnson & Siegler, 1998). Initial knowledge in a domain tends to be conceptual if the target procedure is not demonstrated in the everyday environment or taught in school or if children have frequent experience with relevant concepts before the target procedure is taught. In contrast, initial knowledge generally is procedural if the target procedure is demonstrated frequently before children understand key concepts or if the target procedure is closely analogous to a known procedure in a related domain. Thus, children's prior experience with the domain predicts which type of knowledge sets the learning process in motion. Once children develop some knowledge of one type, the other type of knowledge often begins to develop as well.

Traditional pretest–posttest designs are unable to detect gradual, bidirectional relations between conceptual and procedural knowledge, so in this study we used a microgenetic approach. In microgenetic studies, knowledge is assessed repeatedly during periods of rapid change to infer the processes that gave rise to the change. Past research using microgenetic methods has yielded a more precise understanding of change in those domains than has hitherto been available (e.g., Alibali & Goldin-Meadow, 1993; Kuhn, Schauble, & Garcia-Mila, 1992; Siegler & Crowley, 1991). The use of fine-grained and repeated assessments of conceptual and procedural knowledge allowed us to assess the iterative development of the two types of knowledge.

A Potential Change Mechanism: Improved Problem Representation

The process orientation of the iterative model highlights the need to identify mechanisms underlying the influence of each type of knowledge on the other. Improved problem representation is one pervasive mechanism of cognitive development (Siegler, 1989). We define *problem representation* as the internal depiction or re-creation of a problem in working memory during problem solving. People form a problem representation each time a problem is solved. *Problem representation* refers to this transitory, internal representation of individual problems (Kaplan & Simon, 1990).

How might improved problem representation underlie the relations between conceptual and procedural knowledge? First, it may underlie the link from conceptual knowledge to improved procedural knowledge. Children's conceptual knowledge may guide their attention to relevant features of problems and help them to organize this information in their internal representation of the problems. This well-chosen problem representation may then support generation and use of effective procedures. Second, improved problem representation may underlie the link from procedural knowledge to improved conceptual knowledge. Use of correct procedures could help children represent the key aspects of problems, which could lead to improved conceptual understanding of the domain. In this research we evaluated the first pathway: the link from improved conceptual knowledge to improved problem representation to improved procedural knowledge.

Several lines of research support the hypothesis that forming correct problem representations is one mechanism linking improved conceptual knowledge to improved procedural knowledge. First, amount of conceptual knowledge in a domain is positively correlated with the accuracy and elaborateness of problem representations (Chase & Simon, 1973; Chi, Feltovich, & Glaser, 1981). Second, manipulations that result in improved conceptual knowledge can lead to improved problem representation (Rittle-Johnson & Alibali, 1999). Third, the quality of problem representations is positively correlated with procedural knowledge in that domain (Morales, Shute, & Pellegrino, 1985; Rittle-Johnson & Alibali, 1999; Siegler, 1976; Sternberg & Powell, 1983). Fourth, manipulations that result in improved problem representations also lead to improved procedural knowledge (Alibali, McNeil, & Perrott, 1998; Siegler, 1976). However, the complete pathway from improved conceptual knowledge to improved problem representation to improved procedural knowledge has not been evaluated using a single task or a single sample of participants.

Development of Conceptual and Procedural Knowledge of Decimal Fractions

We examined the iterative development of conceptual and procedural knowledge in children's learning about decimal fractions.[1] Decimal fraction knowledge is a central component of mathematical understanding. Formal instruction regarding decimal fractions begins by the fourth grade and continues throughout middle school. However, children struggle to understand decimal fractions, and some never master them. In the mathematics assessment of the fourth National Assessment of Educational Progress, half of seventh graders held basic misconceptions about decimal fractions (Kouba, Carpenter, & Swafford, 1989), and a substantial number of adults continue to hold such misconceptions (Putt, 1995; Silver, 1986). Interventions that eliminate misconceptions and improve understanding of decimal fractions are greatly needed.

Furthermore, the domain of decimal fractions is a particularly good one for examining the role of representation in the development of conceptual and procedural knowledge because there is a powerful external representation that can be applied to decimal fractions: the number line. Number lines provide an external depiction of key decimal fraction concepts (Hiebert, Wearne, & Taber, 1991; Moss & Case, 1999; NCTM, 1989), and children have been hypothesized to use an internal number line to represent whole numbers (Case & Okamoto, 1996). Thus, highlighting the relevance of the number line to decimal fractions could improve children's representations of them.

Because of the potential power of the number line for representing decimal fractions, we developed an intervention using number line problems. During the intervention, fifth- and sixth-grade children placed decimal fractions on number lines and received feedback. These number line problems are not part of traditional curricula, so most children do not have prior experience with procedures for solving the problems. In contrast, fifth- and sixth-grade students have been exposed to relevant decimal fraction concepts such as place value, magnitude, equivalent values, and the role of zero as a place holder (Hiebert, 1992; Hiebert & Wearne, 1983; Resnick et al., 1989). Thus, we expected children to begin the study with some conceptual knowledge of decimal fractions but little or no procedural knowledge for placing decimal fractions on number lines.

This initial conceptual knowledge was expected to help children form either of two correct representations of decimal fractions. In the *common unit* approach, a decimal fraction is represented in terms of its smallest unit (e.g., hundredths, thousandths). For example, 0.745 can be represented as 745 thousandths. Conceptual understanding of place values and units should be related to formation of such common unit representations. In the alternative, *composite* approach, decimal fractions are represented as the sum of the individual column values. Within this framework, 0.745 would be represented as the sum of 7 tenths, 4 hundredths, and 5 thousandths. Conceptual understanding of place values and additive composition of numbers should help in formation of this type of representation.

Each of these representations is related to a particular procedure for correctly locating decimal fractions on a number line. The composite representation is related to a procedure in which the child roughly divides the number line into tenths and first counts out or estimates the number of tenths from the origin indicated by the digit in the tenths column. The common unit representation is related to a procedure in which the child envisions the magnitude of the decimal fraction relative to the number of units (e.g., 745

[1] *Decimal fraction* is the mathematical term for base-10 numbers that include values that are less than one whole. However, in mathematics instruction textbooks and in everyday language, decimal fractions are simply called *decimals*.

units relative to 1,000) and then translates the result onto a position on the number line.

Thus, children's initial conceptual knowledge of decimal fractions should support learning of correct procedures by means of correct problem representation. In addition, children with greater conceptual knowledge should be more likely to generate meaningful explanations for why the correct answer is correct and how it was generated (Chi, Bassok, Lewis, Reimann, & Glaser, 1989). Generating these explanations should help support greater learning of correct procedures (Pine & Messer, 2000; Siegler, 1995). Thus, we predicted that children with greater pretest conceptual knowledge relevant to decimal fractions would learn more from the instructional intervention than children with less initial conceptual understanding.

Overview of Experiments

To evaluate the iterative model, we assessed children's conceptual and procedural knowledge of decimal fractions before and after a brief instructional intervention. In Experiment 1 we examined individual differences in prior knowledge and in amount of learning; the goal was to provide correlational support for each of the links in the iterative model. In Experiment 2 we experimentally manipulated support for forming correct problem representation during the intervention; the goal was to evaluate the causal link from formation of correct problem representation to improved procedural knowledge.

Experiment 1

We hypothesized that conceptual knowledge of decimal fractions at pretest would predict changes in procedural knowledge for solving number line problems from pretest to posttest. These changes in procedural knowledge, in turn, were expected to predict pretest–posttest changes in conceptual understanding of decimal fractions. We further predicted that the link from initial conceptual knowledge to improved procedural knowledge following the intervention would be explained, at least in part, by formation of correct problem representations.

Method

Participants

Seventy-four students (33 girls and 41 boys) participated near the end of their fifth-grade school year. Their mean age was 11 years, 8 months. The students were drawn from two rural public elementary schools that served a predominantly White population from a range of socioeconomic backgrounds. Both schools used traditional mathematics textbooks. Because of the instructional goals of the experiment, an additional 25 students were excluded from the study because they solved at least two thirds of the procedural knowledge problems correctly at pretest.

Assessments

As discussed in the beginning of this article, a key distinction between assessments of conceptual and procedural knowledge is the novelty of the tasks. To some extent, solving any task relies on the use of procedures (e.g., executing actions), so the distinction is whether children already know a procedure for solving the task or whether they must generate a new procedure to solve it (on the basis of their conceptual knowledge; Greeno et al., 1984). We distinguished between assessments of conceptual and procedural knowledge on the basis of the novelty of the tasks at posttest. Because children received repeated practice and feedback with number line problems during the intervention, this task became familiar and routine and thus tapped children's procedural knowledge. In contrast, the tasks used to assess general fraction ideas, such as equivalent values, were novel and were not presented during the intervention, so these tasks assessed conceptual knowledge. At pretest, the distinction between the conceptual and procedural knowledge tasks was less clear, because children lacked prior experience with any of the tasks before beginning the experiment. Nevertheless, for the sake of consistency we use the label *procedural knowledge* to refer to the knowledge tapped by the number line problems throughout the experiment. Performance on the number line problems at pretest simply provides a baseline for interpreting later performance on the problems.

Procedural knowledge tests. The procedural knowledge assessments measured children's ability to place decimal fractions on number lines. The number line problems are outlined in Table 1 and were presented on four occasions: at pretest (9 problems), during the intervention (12 problems), at posttest (15 problems), and on a transfer test (6 problems). The pretest, posttest, and transfer test problems involved paper-and-pencil presentation and responses; the intervention problems involved computerized presentation and responses.

Table 1
Procedural Knowledge Assessments in Experiment 1: Types of Number Line Problems and Scoring System

Phase	Task	Scoring system
Pretest, intervention, and posttest	Mark the position of a decimal fraction on a number line from 0 to 1 (with tenths marked, as in Figure 2).	Answer within correct tenths section (e.g., 0.87 must be between 0.8 and 0.9).
	Mark the position of a decimal fraction on a number line from 0 to 1 that does not have the tenths marked.	Answer no more than 1 tenth from correct placement (e.g., 0.63 must be between 0.53 and 0.73).
	Choose the decimal fraction for a given position on a number line from 0 to 1 that does not have the tenths marked.	Select correct answer.
Transfer	Mark positions of a pair of decimal fractions on a number line from 0 to 1 that does not have the tenths marked.	Relative order correct and each number no more than 1.5 tenths from correct placement.
	Mark positions of a pair of decimal fractions that are greater than 1 on a number line from 0 to 10 with only the end points marked.	Relative order correct and each number no more than 1.5 units from correct placement.

On the pretest and posttest, the left end of the number line was marked "0," and the right end was marked "1." On one third of problems in each phase, hatch marks divided the number line into 10 equal sections, and children were asked to add a mark to specify the location of a given decimal fraction. On another one third of problems the task was the same, but the number line was not divided into tenths. On the remaining one third of problems children were presented a single hatch mark on the number line and were asked to choose which of four decimal fractions corresponded to it. The target numbers were of one of five types: one, two, or three digits with no zero in the tenths column (e.g., 0.2, 0.87, 0.522) or two- or three-digits with a zero in the tenths column (e.g., 0.09, 0.014).

The problems in the intervention phase were similar except that all of them involved multiple-choice responses. Rather than children marking the number line at any location, they were presented four possible locations and asked to identify the one that corresponded to the decimal fraction presented on that problem. Here, as well as on the multiple-choice problems on the pretest and posttest, the three foils were designed to reflect common incorrect ways of thinking about decimal fraction problems (Resnick et al., 1989). For example, when asked to mark 0.509 on the number line (see Figure 2), foils included a hatch mark at the location corresponding to 0.8, which might be attractive to children who thought numbers with more digits should go toward the large end of the number line; a hatch mark at the location corresponding to 0.15, which might be attractive to children who thought that because thousandths are small pieces, the number should go toward the low end of the number line; and a hatch mark at the location corresponding to 0.05, which might be attractive to children who were confused over the role of zero in the tenths column.

The problems on the transfer test differed in two ways from those presented in the other three phases. On all of these problems, children were asked to mark the locations of two decimal fractions, rather than one, on the number line. In addition, on half of these problems, the number line ranged from 0 to 10, rather than from 0 to 1. None of the number lines on the transfer task included markings other than the numbers at the ends of the line.

Conceptual knowledge test. Understanding of several decimal fraction concepts was assessed with the five paper-and-pencil tasks shown in Table 2. The same tasks were presented on a pretest (before the intervention) and on a posttest (after it). We did not rely on verbal explanations as a measure of conceptual knowledge, because conceptual knowledge can be implicit and because children sometimes have difficulty articulating their knowledge (Bisanz & LeFevre, 1992; Brainerd, 1973; Greeno & Riley, 1987). Several of our tasks were adapted from conceptual knowledge assessments used by Hiebert and Wearne (1983) and Resnick et al. (1989). Four of the five tasks were novel to the participating children, based on examination of their textbooks. The fifth task, the relative magnitude task, was presented during one lesson in the textbook; however, we included this task on the assessment because many past studies on conceptual understanding of decimal fractions have used this task and have found that the problems tap children's misconceptions of decimal magnitude, even after direct classroom instruction (Ellis, Klahr, & Siegler, 1993; Moloney &

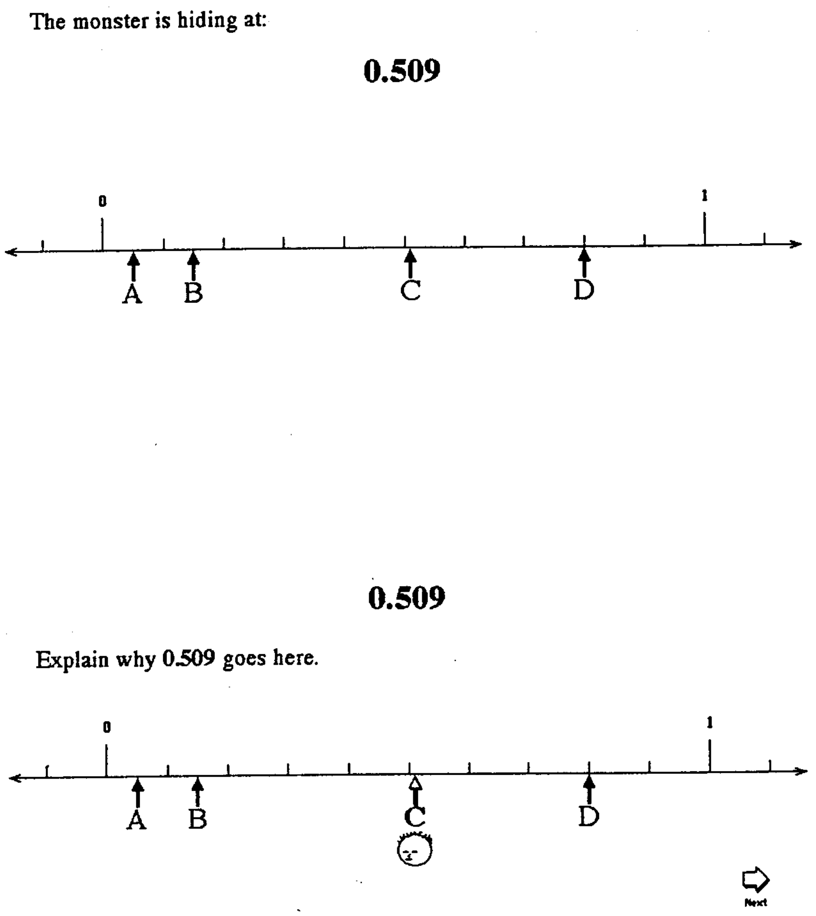

Figure 2. Catch the Monster game used in the intervention.

Table 2
Conceptual Assessment of Decimal Fraction Knowledge in Experiment 1

Concept	Task and scoring criteria
Relative magnitude	Circle the larger of two decimal fractions (n = 8; credit given for number correct over 4)
Relations to fixed values	Choose decimal fraction that is nearest to, greater than, or less than the target (n = 4)
Continuous quantities	Write a number that comes between decimal fractions A and B (n = 4)
Equivalent values (& zero as placeholder)	Circle all the numbers that are equivalent to a given decimal fraction (n = 2; 2 correct per question)
Plausible addition solutions	Evaluate correctness of possible answers to decimal fraction addition problems (n = 2)

Note. On the eight relative magnitude problems, four problems could be solved correctly using an incorrect rule, such as choosing the longer sequence of digits, or by guessing. Therefore, children received credit only for each problem they solved correctly over four (for a maximum of 4 points).

Stacey, 1997; Putt, 1995; Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985).

Computer Program for Problem-Solving Intervention

During the instructional intervention, children played a computer game that we developed called "Catch the Monster." The three types of number line problems outlined above were presented during the game. For the first two problem types children were presented a decimal fraction and a number line with arrows pointing to four locations (see Figure 2). Each arrow represented a location where the monster could be hiding; the decimal fraction indicated the monster's actual location. The child's task was to identify the correct arrow for the given decimal fraction. On the third problem type, the monster appeared under a single hatch mark that crossed the number line; the child's task on these trials was to determine which of four decimal fractions corresponded to the monster's location.

Before children played the game, the experimenter demonstrated on two sample problems what the child needed to do to play the game. Children then were presented 12 problems to solve on their own. The experimenter left the room during this part of the intervention, because children learn more during this task when the experimenter is not present (Rittle-Johnson & Russo, 1999). After each answer, the computer program provided feedback on the monster's location (i.e., the correct answer) and prompted the child to explain why that answer was correct (see Figure 2). Children's oral responses were recorded with a tape recorder. The computer program was written in HyperCard 2.3 and was presented on a Powerbook laptop computer.

Procedure

Children completed the conceptual and procedural knowledge pretests in their classrooms. Each child subsequently participated in an individual intervention session that lasted approximately 40 min.

To increase the likelihood that children would learn, they were presented one of four brief lessons at the beginning of the intervention session. The instruction focused on either conceptual knowledge relevant to the problems, a procedure for solving the problems, both, or neither. Children in all four groups showed comparable patterns of learning, so the instructional manipulation is not considered further.

All children then played the Catch the Monster game. After they finished, they were presented the conceptual knowledge posttest, the procedural knowledge posttest, and the transfer test. The session was videotaped.

Coding

Procedural knowledge. On the pretest, posttest, and transfer test, each child's mark on each number line was translated into the corresponding number to the nearest hundredth. The accuracy criterion for each problem type is presented in Table 1. The intervention problems were multiple choice, so choice of the correct answer was used to assess accuracy on these problems. Percentage correct on each assessment indexed procedural knowledge in that phase.

Conceptual knowledge. Children received 1 point for each question answered correctly on the conceptual knowledge assessment at pretest and posttest, with a maximum of 18 points at each time (see Table 2). Each child also received a *conceptual improvement* score, which indexed change in children's conceptual knowledge from pretest to posttest, relative to the amount of possible improvement for that child. Conceptual improvement was defined as: (number correct at posttest − number correct at pretest) ÷ (18 − number correct at pretest).

Problem representations. Children's ability to represent the problems correctly was assessed during the intervention by means of their explanations of the correct answers. As discussed in the beginning of this article, there were two correct ways to represent decimal fractions. Composite representations were inferred when children generated explanations such as (for 0.70): "There are 7 tenths, little lines" or "You count over 7 to the line, and the 0 means nothing is in the hundredths." Common unit representations were inferred from explanations such as (for 0.025): "It's divided into thousandths, and you count 25 thousandths." Explanations that did not reflect correct problem representations sometimes reflected reliance on analogies to whole numbers. For example, for the target value 0.509, one child explained: "It is in the hundreds and it starts with a 5."

For each child, correct problem representation was indexed by the percentage of intervention problems represented correctly, using either of the two correct approaches. Two raters independently coded every explanation for whether the child represented the problem correctly and, if so, which form of representation was used. The two raters agreed on 92% of trials, both about whether the representation was correct and, if it was, which correct representation was used. They agreed on whether the representation was correct on 98% of trials.

Results and Discussion

The results and discussion are organized around the iterative model. First, we provide an overview of learning outcomes. Next, we examine the bidirectional, iterative relations between conceptual and procedural knowledge. Finally, we explore the role of problem representation as a link from initial conceptual knowledge to improved procedural knowledge following instruction. All reported results are significant at the .05 level, unless otherwise noted. There were no effects for gender in any analysis.

Overview of Learning Outcomes

Knowledge at pretest. At pretest, children had some knowledge of decimal fraction concepts (see Table 3). They answered 33% of the questions on the conceptual knowledge assessment correctly. Children also solved 28% of the number line problems

Table 3
Percentage Correct on Each Subtask on the Conceptual Knowledge Assessment in Experiment 1

Test	Relative magnitude	Relations to fixed values	Continuous quantities	Equivalent values	Plausible addition solutions
Pretest	19	21	34	54	41
Posttest	32*	40*	50*	56	47

* $p < .05$, improvement from pretest to posttest, based on paired t tests.

correctly at pretest. Because of their lack of previous experience with the number line problems, the procedural knowledge pretest likely did not tap prior knowledge of specific procedures for performing the task but rather a combination of procedures imported from other domains, conceptual knowledge, and guessing. At least some children seemed to use a whole number approach to solve the problems. They focused on the number of digits in the target and ignored zeros to the left of the number. For example, 39% of children followed a "more digits means a bigger number" approach by marking a 3-digit number more than 2 tenths higher than its actual position. Similarly, 40% of children ignored zeros when they were in the tenths position (e.g., marking 0.07 as 0.7). This whole number approach sometimes led to the correct answer. For example, 65% of children correctly marked 0.2 on the number line, which follows the logic of whole numbers. In addition to importing procedures from the domain of whole numbers, children may have relied on their conceptual knowledge of the domain to solve the number line problems, which were novel at pretest. Success on the procedural and conceptual knowledge pretests was correlated, $r(72) = .33$, suggesting some overlap in the knowledge tapped by the two assessments. Finally, children could get a few problems correct by chance. On the free response items, the range of acceptable positions for a given number was 2 tenths out of a range of 10 tenths, making chance performance 20%. On the multiple-choice items chance performance was 25% correct. The percentage of problems children solved correctly (28%) was not much higher than these percentages. Overall, children had some success solving number line problems correctly at pretest, but this success was unlikely to result from prior procedural knowledge for locating decimal fractions on number lines.

Improved conceptual knowledge. Although children had no experience during the intervention with the tasks on the conceptual knowledge assessment, their conceptual knowledge of decimal fractions was higher on the posttest than on the pretest (45% vs. 33% correct), $t(73) = 6.32$, $\eta_p^2 = 0.35$. These improvements in conceptual knowledge were observed on three of the five tasks on the conceptual assessment (see Table 3). One reason for this improvement was that children less often misapplied whole number knowledge on the posttest. For example, on the relative-magnitude task on the pretest, 53% of the children treated decimal fractions like whole numbers by always choosing the number with more digits as the bigger number. At posttest, only 30% of children used this whole number approach, McNemar's test, $\chi^2(1, N = 74) = 2.96$, $p = .08$. Alongside this general improvement, individual differences in conceptual knowledge proved highly stable from pretest to posttest, $r(72) = .77$.

Improved procedural knowledge. Children also learned correct procedures for solving number line problems over the course of the study. Compared to solving 28% of problems on the pretest correctly, children solved considerably more problems correctly during the intervention phase ($M = 60\%$), $t(73) = 10.41$, $\eta_p^2 = 0.60$, and on the posttest ($M = 68\%$), $t(73) = 14.16$, $\eta_p^2 = 0.73$. They also solved 47% of transfer problems correctly, for which no equivalent problems had been presented on the pretest. Unlike the conceptual knowledge pretest, which strongly predicted posttest conceptual knowledge scores, percentage correct answers on the procedural knowledge pretest was only modestly related to success on the intervention, procedural knowledge posttest, and transfer test, $rs(72) = .33$, 31, and .30, respectively. In contrast, percentage correct answers on the intervention was far more predictive of success on the posttest and transfer tests, $rs(72) = .81$ and .68, respectively. Performance on the procedural knowledge pretest did not seem to be based on the same type of knowledge as performance on the later assessments of procedural knowledge. Nevertheless, the procedural knowledge pretest provided a baseline for each child for interpreting performance on later assessments.

Iterative Relations Between Conceptual and Procedural Knowledge

Prior conceptual knowledge of decimal fractions was expected to predict improvements in procedural knowledge from the pretest to the later phases of the experiment. These gains in procedural knowledge, in turn, were expected to predict improvements in children's conceptual knowledge from pretest to posttest.

Initial conceptual knowledge → gains in procedural knowledge. Success on the procedural knowledge pretest was entered first as a control variable in all regression analyses to adjust for the effectiveness of children's initial attempts to solve the problems. We then examined whether pretest conceptual knowledge was related to procedural knowledge in the intervention, posttest, and transfer phases. Initially, the three learning assessments were entered as a single within-subject variable having three levels (intervention, posttest, and transfer test). Later, we examined them separately.

As predicted, the conceptual knowledge pretest was a significant predictor of overall procedural knowledge gain, $F(1, 71) = 34.51$, $\eta_p^2 = 0.33$. There was also an interaction between the conceptual knowledge pretest and the learning assessment, $F(2, 142) = 3.94$, $\eta_p^2 = 0.05$, indicating that the influence of prior conceptual knowledge was not equivalent on the intervention, posttest, and transfer tests. To interpret this interaction, we conducted separate regression analyses for the three phases. After controlling for percentage correct on the procedural knowledge pretest, percentage correct on the conceptual knowledge pretest accounted for 23% of the variance in performance on the procedural knowledge

Table 4
Frequency of Correct Problem Representation as a Mediator Between the Conceptual Knowledge Pretest and Procedural Knowledge on the Intervention, Posttest, and Transfer Test

	Intervention			Posttest			Transfer test		
Step	Partial r^2	β	F^a	Partial r^2	β	F^a	Partial r^2	β	F^a
Step 2									
Conceptual pretest	.23	.51	24.71	.20	.48	20.53	.28	.56	31.90
Step 3									
Conceptual pretest	.07	.30	10.65	.07	.30	8.68	.13	.43	17.93
Representation	.21	.51	32.44	.15	.44	19.78	.09	.34	11.74

[a] For step 2, df = 1, 71; for step 3, df = 1, 70.

tested during the intervention, $F(1, 71) = 24.71$, 20% of the variance at posttest, $F(1, 71) = 20.52$, and 28% of the variance on the transfer test, $F(1, 71) = 31.90$. Thus, prior conceptual knowledge predicted generation, maintenance, and transfer of correct procedures. It was most influential for transfer of correct procedures to novel problems.

In contrast, once the conceptual knowledge pretest was added to each model, the procedural knowledge pretest was not a significant predictor of success on any assessment. The number line pretest may not have tapped procedural knowledge, because children did not have prior experience with the problems. Instead, scores on the procedural knowledge pretest were related to gains in procedural knowledge only to the extent that both were related to initial conceptual knowledge.

Procedural knowledge → gains in conceptual knowledge. On the basis of the iterative model, procedural knowledge acquired during the intervention phase should predict pretest–posttest improvements in conceptual knowledge. Controlling for scores on the procedural knowledge pretest, procedural knowledge scores on the intervention accounted for 22% of the variance in improvement in conceptual knowledge from pretest to posttest, $F(1, 70) = 21.31$.[2] The procedural knowledge posttest and transfer tests were similarly related to pretest–posttest improvements in conceptual knowledge ($\Delta R^2 = .23$), $F(1, 70) = 22.85$, ($\Delta R^2 = .21$), $F(1, 70) = 24.98$, respectively.

These results are consistent with the iterative model of the development of conceptual and procedural knowledge. Children's pretest conceptual knowledge predicted learning of correct procedures, and learning of correct procedures predicted further improvements in conceptual knowledge.

Problem Representation as a Link From Initial Conceptual Knowledge to Improved Procedural Knowledge

To examine the role of correct problem representation in the change process we first examined whether pretest conceptual knowledge predicted correct problem representation during the intervention phase, then examined whether correct problem representation predicted gains in procedural knowledge from the pretest to subsequent phases, and then examined whether correct problem representation mediated the relation between pretest conceptual knowledge and subsequent procedural knowledge. Finally, we considered whether the particular type of correct problem representation that children generated influenced the likelihood of learning correct procedures.

Conceptual knowledge → problem representation. Children represented 57% of intervention problems in one of the two correct ways (composite or common unit). Children's pretest conceptual knowledge scores accounted for 19% of the variance in the percentage of intervention problems that children represented correctly, $F(1, 72) = 16.80$.

Problem representation → gains in procedural knowledge. To examine the influence of correct problem representation on procedural knowledge gain, we conducted regression analyses similar to those reported above. In these analyses, percentage of intervention problems represented correctly by the child was the predictor variable. As expected, children's frequency of correct problem representation predicted their overall procedural knowledge gain, $F(1, 71) = 46.47$, $\eta_{p^2} = 0.40$. Frequency of correct problem representation during the intervention predicted percentage correct answers on the intervention problems, $\Delta R^2 = .37$, $F(1, 71) = 50.20$; posttest problems, $\Delta R^2 = .29$, $F(1, 71) = 32.96$; and transfer problems, $\Delta R^2 = .23$, $F(1, 71) = 24.87$, after controlling for the procedural knowledge pretest.

Mediation analyses. We conducted mediation analyses to explore whether the relation between initial conceptual knowledge and improved procedural knowledge might be explained by improvements in problem representation. If correct problem representation mediated the relation between pretest conceptual knowledge and gains in procedural knowledge, the relation between pretest conceptual knowledge and subsequent procedural knowledge should be substantially reduced when frequency of correct problem representation is included in the regression equation (Baron & Kenny, 1986). To explore this hypothesis, we conducted separate hierarchical regression analyses using each assessment of procedural knowledge beyond the pretest as a dependent variable. In the first step of each analysis the procedural knowledge pretest was entered as a control variable. In the second step, conceptual knowledge at pretest was entered. In the third step, percentage of intervention problems represented correctly was entered. Comparing the influence of pretest conceptual knowledge on procedural knowledge at Steps 2 and 3 indicated the degree to which the

[2] One child was excluded from these analyses because he was at ceiling on the conceptual knowledge pretest.

influence of prior conceptual knowledge was attenuated when correct problem representation was included in the model.

The outcomes of the mediation analyses are presented in Table 4. The conceptual knowledge pretest initially accounted for about 25% of the variance in percentage correct on each assessment of procedural knowledge gain (Step 2). After frequency of correct problem representation was entered into the equation (Step 3), prior conceptual knowledge accounted for a much smaller, but still significant, portion of the variance. Entering frequency of correct problem representation resulted in a 54%–70% reduction in the variance accounted for by prior conceptual knowledge on the three assessments. Thus, the statistical relation between initial conceptual knowledge and improved procedural knowledge is partially accounted for by the intermediary step of improved problem representation. Although other mechanisms may also underlie this link, improved problem representation is one promising mechanism.

Form of correct representation. Correct problem representation predicted acquisition of procedural knowledge. As discussed at the beginning of this article, there are two correct ways to represent decimal values: a composite representation and a common unit representation. The composite representation was more frequent (36% of intervention phase trials), but the common unit representation was also used fairly often (17% of trials).

After controlling for scores on the procedural knowledge pretest, frequency of composite representations accounted for a significant portion of the variance in percentage correct on the intervention, $\Delta R^2 = .31$, $F(1, 71) = 38.68$; posttest, $\Delta R^2 = .25$, $F(1, 71) = 27.39$; and transfer problems, $\Delta R^2 = .23$, $F(1, 71) = 23.30$. In contrast, separate analyses indicated that frequency of common unit representations did not account for a significant portion of the variance in any assessment of procedural knowledge.

To summarize, correctly representing the value of decimal fractions, and in particular forming a composite representation of their magnitude, may help to explain the link between initial conceptual knowledge and improved procedural knowledge. This is consistent with our hypothesis that improved problem representation is one mechanism of change in acquiring procedural knowledge.

Experiment 2

Experiment 1 provided correlational evidence for the relation between correct problem representation and development of procedural knowledge. The primary goal of Experiment 2 was to provide causal evidence for this relation.

We experimentally manipulated the likelihood that children would form correct problem representations during an intervention, using two techniques guided by observations of successful learners in Experiment 1. One manipulation involved providing prompts to notice the tenths digit in the target number. This manipulation was based on the finding from Experiment 1 that successful learners tended to note the value of this digit in their explanations (i.e., to use composite representations). The second manipulation involved presenting number lines that were divided into 10 equal sections. Again, the purpose was to promote use of the composite representation, in this case by illustrating the meaning of "tenths" in the context of the number line, thereby facilitating the mapping between numerical and spatial representations of the decimal fraction. Thus, the first manipulation promoted formulation of a composite representation of the decimal fraction; the second manipulation promoted formulation of a composite representation of the number line. We predicted that both manipulations would lead to improved problem representation and thus to improvements in procedural knowledge.

Method

Participants

Fifty-nine fifth-graders (33 girls and 26 boys) and 58 sixth-graders (28 girls and 30 boys) participated during their fall semester. The fifth-graders' mean age was 10 years, 6 months; the sixth-graders' was 11 years, 6 months. The students attended one of two parochial schools located in a predominantly White urban or suburban neighborhood. Both schools used traditional mathematics textbooks. Children completed the Mathematics subtest of the Iowa Test of Basic Skills as part of this study. On average, the fifth graders scored in the 63rd percentile, and the sixth graders scored in the 50th percentile. An additional 9 students (2 fifth-graders and 7 sixth-graders) were excluded from the study because they already knew how to solve the number line problems at pretest (i.e., they solved at least 90% of the problems correctly).

Assessments

Procedural knowledge. The procedural knowledge assessments were similar to those used in Experiment 1. The 10 problems on the pretest and posttest were all of the second type listed in Table 1. This type of problem was also used during the intervention phase, with the modification that the tenths were marked for children in the tenths-marked conditions. The third type of problem listed in Table 1 was moved to the transfer test, and one of the other types of transfer problems was revised so that the digit in the tenths column was the same for both numbers (e.g., mark 0.46 and 0.497).

Conceptual knowledge. The conceptual knowledge assessment was identical to that used in Experiment 1, except that one task was replaced. The task on which children were asked to evaluate plausible addition answers was removed, because children showed little change on it following the intervention and because it was not directly related to understanding of decimal fractions. It was replaced with a task about understanding of place value. On the four questions of this task, children were asked to identify the digit in the tenths or hundredths position of a given number and to decide if adding a zero in the tenths column would influence the number's value. This resulted in a conceptual knowledge assessment containing 5 tasks worth 4 points each, for a total of 20 possible points.

Representation. Three measures of problem representation were used in this experiment. The first was based on children's explanations of the correct answer on the intervention problems, as in Experiment 1. Two new assessments also were piloted in this experiment. One was an encoding task, in which children were shown decimal fractions for 5 s and then were asked to write the numbers exactly as they had seen them. The other was a recognition task, in which children were asked to identify the numbers they had just placed on the number lines. Children were given sets of four numbers and were asked to circle the number in each set that they remembered placing on the number line. This assessment was given after each procedural knowledge assessment.

Individual-difference measures. We also assessed individual differences in mathematics achievement and motivation in this experiment. We used performance on the Mathematics subtest of the Iowa Test of Basic Skills—Survey Battery (Form M) to assess general achievement. Children were given the level of achievement test appropriate for their grade (either Level 11 or Level 12), and the raw scores were converted to standardized scores. To index motivation, two measures were administered. The first assessed children's learning and performance goals and was based on a questionnaire used by Stipek and Gralinski (1996). The other was taken from the 1986 National Assessment of Educational Progress (Dossey, Mullis, Lindquist, & Chambers, 1988). Children were asked to rate their liking of mathematics (e.g., "I enjoy mathematics") and beliefs about mathematics (e.g., "Learning mathematics is mostly memorizing"). Both of the motivation assessments used a 5-point rating scale.

Computer Program for Intervention

The Catch the Monster game used in Experiment 1 was again used. The program was adapted slightly to implement the two manipulations of representational support. First, children in the *prompted* conditions heard one of three randomly selected prompts as each trial was presented: "Notice the first digit," "Don't forget to notice," or "Remember the first digit." The first digit after the decimal point was also highlighted in red throughout each trial. Second, children in the *tenths marked* conditions saw number lines divided into 10 equal sections by hatch marks, whereas the other children saw number lines that did not have the tenths marked.

Procedure

Children first completed the conceptual and procedural knowledge pretests, the two new measures of problem representation, and the mathematics motivation assessments in their classrooms. In a separate classroom session children completed the assessment of mathematics achievement.

The remainder of the experiment was conducted individually for each child in sessions of approximately 40 min. Participants were randomly assigned to one of four intervention conditions that varied in two forms of representational support: (a) prompts to notice the first digit along with number lines marked with 10 sections ($n = 30$), (b) prompts only ($n = 29$), (c) marked number lines only ($n = 27$), or (d) neither form of support (control; $n = 31$).

After solving three warm-up problems with paper and pencil, and one warm-up problem on the computer, children were presented the 15 Catch the Monster problems. Before solving the problems, children in the prompts conditions were told: "You should think about the first digit after the decimal point before you pick your answer. Don't ignore the other digits, but pay particular attention to the first digit after the decimal point." For children who were in the tenths marked conditions, the fact that the number line was divided into 10 sections was mentioned in the instructions. The term *tenths* was never used by the experimenter. On each problem, children selected an answer, received feedback on the correct answer, and were prompted to explain why the number should be placed at that (correct) position on the number line. As in Experiment 1, the experimenter was not in the room during the intervention phase.

After the children had completed the intervention problems, the experimenter returned, and children took the procedural knowledge posttest, the conceptual knowledge posttest, and the transfer test.

Coding

Coding of conceptual knowledge, procedural knowledge, and problem representation during the intervention was the same as in Experiment 1. Two independent raters coded the problem representations of 20% of the children. The two raters agreed on 83% of trials, both about whether the representations was correct and, if it was, which correct representation was used. They agreed on whether the representation was correct on 84% of trials.

The two new assessments of children's representations did not serve their intended purposes. Children were already at ceiling on the pretest on the encoding task, so the measure could not be used to assess change. Performance on the recognition task did not correlate significantly with other measures of problem representation, which suggested that it did not assess what it was intended to measure. Thus, these two assessments will not be considered further.

Results and Discussion

We present the results in three sections. First, we provide an overview of children's learning over the course of the study. Next, we examine the effects of the manipulations of representational support on the formation of problem representations and on improvements in procedural knowledge. Finally, we test the iterative relations between conceptual and procedural knowledge described in Figure 1. All reported results are significant at the .05 level, unless otherwise noted. There were no effects of grade or gender on any assessment.

Overview of Learning Outcomes

Pretest performance. As in Experiment 1, children began the study with some conceptual knowledge of decimal fractions (M = 40% correct, see Table 5). They also solved 34% of problems correctly on the procedural knowledge pretest. As in Experiment 1, correct answers on the procedural knowledge pretest seemed to derive from procedures imported from other domains, translation of conceptual knowledge into novel procedures, and guessing. Again, some children seemed to use a whole number approach to solving the problems. Children tended to mark 3-digit decimal fractions closer to the high end of the scale than their actual position (a mean of 1.0 tenth higher) and to mark 1-digit decimal fractions closer to zero than their actual position (a mean of 3.2 tenths lower). Children also tended to ignore zero in the tenths position, thus marking these numbers an average of 3.0 tenths higher than their true location. Children also may have used their knowledge of domain concepts to solve the problems; percentage correct on the conceptual and procedural knowledge pretests was moderately correlated, $r(115) = .45$. Finally, children could solve 20% of the problems correctly by chance.

Improvement in conceptual knowledge. Percentage correct on the conceptual knowledge assessment was higher at posttest than at pretest, (Ms = 51% vs. 40%), $t(116) = 7.22$, $\eta_p^2 = 0.31$. As shown in Table 5, these gains in conceptual knowledge were found

Table 5
Percentage Correct on Each Subtask on the Conceptual Knowledge Assessment in Experiment 2

Test	Relative magnitude	Relations to fixed values	Continuous quantities	Equivalent values	Place value
Pretest	35	26	40	41	59
Posttest	51*	40*	48*	46*	69*

* $p < .05$, improvement from pretest to posttest, based on paired t tests.

on each task on the conceptual knowledge assessment. Another reflection of this increased conceptual understanding was that fewer children used a whole number approach on the relative magnitude task at posttest than at pretest (30% vs. 45%), McNemar's test, $\chi^2(1, N = 117) = 8.49$.

Improved procedural knowledge. Over the course of the study, many children learned correct procedures for solving number line problems. After correctly solving 34% of the pretest problems, children correctly solved 63% of the intervention problems, $t(116) = 10.94$, $\eta_p^2 = 0.51$, and 58% of the posttest problems, $t(116) = 9.01$, $\eta_p^2 = 0.41$. They also correctly solved 41% of the transfer problems, which were not included on the pretest.

Impact of mathematical achievement and motivation. Math achievement scores were related to success on the conceptual and procedural knowledge pretests ($rs[115] = .57$ and $.49$, respectively). However, math achievement scores did not predict success on the intervention, posttests, or transfer test, after controlling for scores on the conceptual and procedural knowledge pretests. The measures of motivation were not related to decimal fraction knowledge on any assessment.

Effects of Experimental Manipulations on Problem Representation and Procedural Knowledge

The manipulations of representational support were expected to aid correct problem representation and to lead to improvements in procedural knowledge. In all analyses, conceptual and procedural knowledge pretest scores were entered first to control for the effects of prior knowledge.

Impact of experimental manipulations on problem representation. The purpose of prompting children to notice the digit in the tenths column and of having the tenths marked on the number line was to facilitate correct problem representation. We conducted a 2 (prompts: present or absent) × 2 (tenths marked: present or absent) analysis of covariance (ANCOVA) on percentage of intervention problems represented correctly. There was a main effect of prompts, $F(1, 111) = 29.95$, $\eta_p^2 = 0.21$, and of tenths markings, $F(1, 111) = 36.91$, $\eta_p^2 = 0.25$. As shown in Figure 3, both manipulations of representational support aided correct problem representation.

Impact of experimental manipulations on acquisition of procedural knowledge. Receiving prompts and having the tenths marked were expected to influence acquisition of procedural knowledge across the intervention, posttest, and transfer test. We conducted a 2 (prompts) × 2 (tenths marked) × 3 (learning assessment: intervention, posttest, or transfer test) ANCOVA on percentage of procedural knowledge items solved correctly. Prompts and tenths marked were between-subjects factors; learning assessment was a within-subject factor.

Receiving prompts, $F(1, 111) = 13.40$, $\eta_p^2 = 0.11$, and having the tenths marked, $F(1, 111) = 9.32$, $\eta_p^2 = 0.08$, both raised the percentage of correct answers, after controlling for scores on the conceptual and procedural knowledge pretests. There was also a significant interaction between the two manipulations. Receiving both forms of representational support led to larger gains in procedural knowledge than would have been expected from each form independently, $F(1, 111) = 4.47$, $\eta_p^2 = 0.04$.

There was also a main effect of learning assessment and an

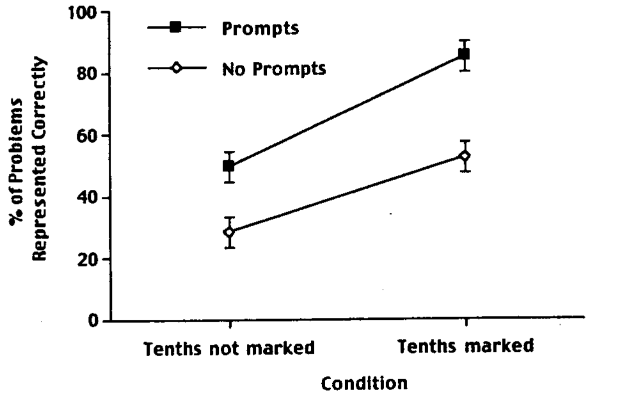

Figure 3. Average percentage correct problem representation during the intervention, by condition.

interaction between learning assessment and each of the two manipulations. As shown in Figure 4, procedural knowledge assessment interacted with receiving prompts, $F(2, 222) = 12.44$, $\eta_p^2 = 0.10$, and with having the tenths marked, $F(2, 222) = 6.56$, $\eta_p^2 = 0.06$. Orthogonal planned comparisons indicated that the main effects of receiving prompts and of having the tenths marked were greater during the intervention than on the posttest and transfer test, $Fs(1, 111) = 21.38$ and 11.01, $\eta_p^2 = 0.16$ and 0.09, respectively, but did not differ between the posttest and transfer test. There were no three-way interactions, indicating that the added benefit of receiving both forms of representational support was similar on all three assessments.

The effects of the manipulations on acquisition of procedural knowledge also were moderated by individual differences in prior conceptual knowledge. The experimental manipulations used in this experiment were based on what children with high conceptual knowledge did spontaneously in Experiment 1. Thus, children with relatively high conceptual knowledge at pretest seemed less likely to benefit from the experimental manipulations, compared to children with relatively low prior knowledge. If this were the case, then the effects of the manipulations should interact with children's prior conceptual knowledge (Baron & Kenny, 1986; Judd & McClelland, 1989). To test for this moderating role, terms for each potential interaction between the conceptual knowledge pretest and the experimental manipulations were added to the initial ANCOVA model.

As expected, there was an interaction between pretest conceptual knowledge and receiving prompts, $F(1, 108) = 5.04$, $\eta_p^2 = 0.04$, and a trend toward an interaction between pretest conceptual knowledge and having the tenths marked, $F(1, 108) = 3.47$), $p = .06$, $\eta_p^2 = 0.03$. To interpret these interactions, we graphed the predicted relation between prior conceptual knowledge and acquisition of procedural knowledge separately for children in each condition (as suggested by Baron & Kenny, 1986). As shown in Figure 5, as prior conceptual knowledge increased, the effects of the manipulations decreased. Children who began with low conceptual knowledge benefited from the representational supports more than children who began with relatively high conceptual knowledge.

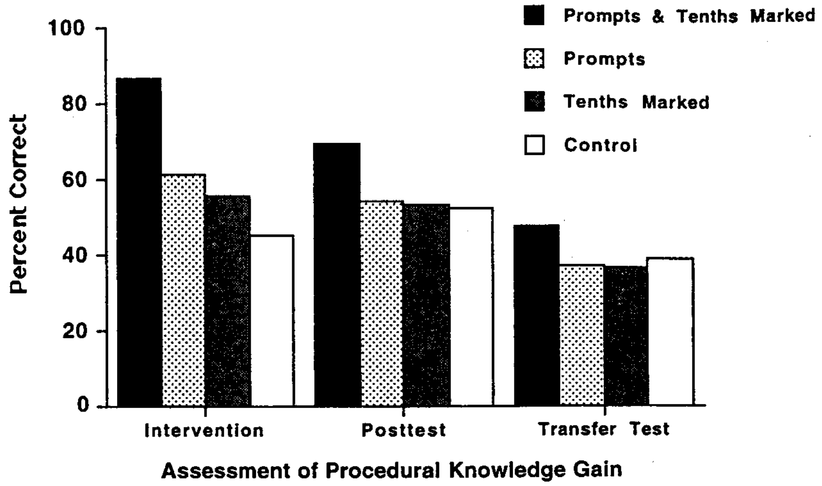

Figure 4. Average percentage correct on each assessment of procedural knowledge gain, by condition.

Iterative Relations Between Conceptual and Procedural Knowledge

These results from Experiment 2 provide evidence for causal links from improved problem representation to improved procedural knowledge. The Experiment 2 results were also expected to replicate the iterative relations between conceptual and procedural knowledge found in Experiment 1.

Conceptual knowledge → gains in procedural knowledge. According to the iterative model, amount of prior conceptual knowledge should be positively related to amount of procedural knowledge acquired. After controlling for scores on the procedural knowledge pretest and experimental condition with ANCOVA, there was a main effect of pretest conceptual knowledge on percentage correct number line placements (the measure of procedural knowledge), $F(1, 111) = 42.43$, $\eta_p^2 = 0.28$.

There was also an interaction between pretest conceptual knowledge and procedural knowledge assessment, $F(2, 222) = 19.94$, $\eta_p^2 = 0.15$. The influence of prior conceptual knowledge differed across the intervention, posttest, and transfer test. In separate regression analyses, scores on the conceptual knowledge pretest had the largest influence on performance on the transfer test, $\Delta R^2 = .32$, $F(1, 111) = 83.61$; and a smaller influence on performance during the intervention, $\Delta R^2 = .08$, $F(1, 111) = 19.79$; and on the posttest, $\Delta R^2 = .06$, $F(1, 111) = 9.23$. Thus, as in Experiment 1, prior conceptual knowledge predicted use of correct procedures on all three learning assessments, but it had the greatest influence on transfer of procedures to novel problems.

Procedural knowledge → gains in conceptual knowledge. According to the iterative model, amount of procedural knowledge should predict improvements in conceptual knowledge from pretest to posttest. We conducted regression analyses to examine whether percentage correct on the intervention problems (a measure of procedural knowledge) predicted pretest to posttest improvement in conceptual knowledge.[3] After controlling for the influence of the procedural knowledge pretest, percentage correct during the intervention accounted for 6% of the variance in conceptual improvement from pretest to posttest, $F(1, 111) = 7.36$. Procedural knowledge on the posttest and transfer test were also related to conceptual improvement: $\Delta R^2 = .12$, $F(1, 111) = 17.56$, and $\Delta R^2 = .37$, $F(1, 111) = 79.3$, respectively. The experimental manipulations did not predict amount of conceptual improvement, after controlling for the differences in percentage correct answers on the intervention problems.

General Discussion

We examined the development of conceptual and procedural knowledge of decimal fractions and the role of problem representation in this development. In both experiments, children's initial conceptual knowledge predicted gains in procedural knowledge, and the gains in procedural knowledge predicted improvements in conceptual knowledge. Correct problem representation was an important link between conceptual and procedural knowledge. In Experiment 1, problem representation partially mediated the link from initial conceptual knowledge to gains in procedural knowledge from pretest to posttest. In Experiment 2, experimental manipulations led to better problem representations and to greater improvements in procedural knowledge. Thus, the results of both experiments supported the iterative model of the development of conceptual and procedural knowledge depicted in Figure 1.

The discussion of these results is organized around three issues: (a) developmental relations between conceptual and procedural knowledge, (b) mechanisms underlying the relations between conceptual and procedural knowledge, and (c) educational implications of the findings.

Relations Between Conceptual and Procedural Knowledge

Conceptual and procedural knowledge did not develop in an all-or-none fashion, with acquisition of one type of knowledge strictly preceding the other. Neither type of knowledge was fully developed at the beginning or at the end of the study; rather,

[3] Three children were excluded from this analysis because they were at ceiling on the conceptual knowledge pretest.

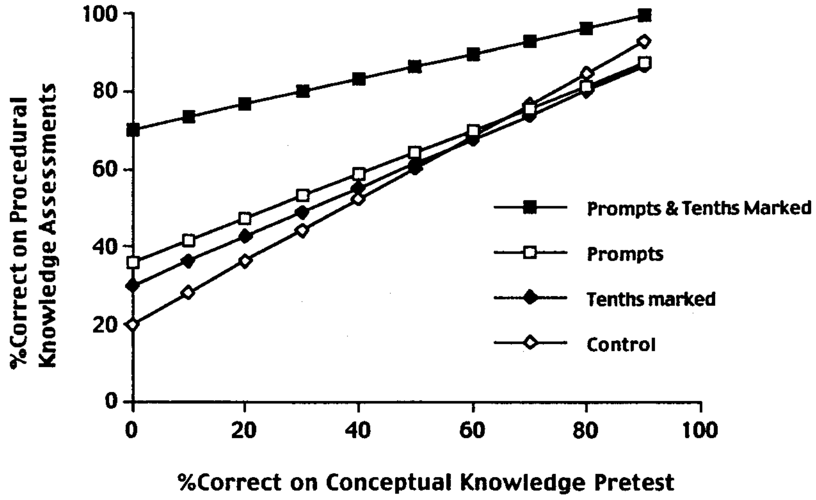

Figure 5. Predicted values for accuracy on the procedural knowledge assessments by prior conceptual knowledge for children in each condition. Effects of condition were moderated by prior conceptual knowledge.

conceptual and procedural knowledge appeared to develop in a gradual, hand-over-hand process. Causal, bidirectional relations between conceptual and procedural knowledge (Rittle-Johnson & Alibali, 1999) may lead to the iterative development of the two types of knowledge. These iterative relations highlight the importance of examining conceptual and procedural knowledge together. Studying one type of knowledge in isolation may lead to an incomplete picture of knowledge change and may obscure important change processes.

On the basis of our findings we argue that the concepts-first versus procedures-first debate is misguided. Claims about one type of knowledge preceding the other are often based on one-shot, dichotomous knowledge assessments and arbitrary criteria for what it means to "have" each type of knowledge. To avoid these pitfalls, in this research we used a more microgenetic approach (Siegler & Crowley, 1991). Multifaceted, continuous measures of knowledge were administered before, during, and after an intervention. This methodology allowed us to detect early, incomplete knowledge states and to chart the iterative, bidirectional development of conceptual and procedural knowledge.

The iterative model of the development of conceptual and procedural knowledge also helps to resolve two issues raised by previous research. First, early knowledge tends to be very limited, making it unclear whether a given behavior indicates "understanding" of a concept or problem-solving procedure. The iterative model explicitly recognizes these partial knowledge states, thus acknowledging the knowledge that children possess without overstating it. Second, early knowledge in a domain can be conceptual or procedural, and prior experience in the domain is likely to determine which type of knowledge begins to emerge first (Rittle-Johnson & Siegler, 1998). In this study, children had previous classroom experience with decimal fraction concepts but not with procedures for placing decimal fractions on number lines. Therefore, initial knowledge in the domain was conceptual, and this conceptual knowledge facilitated learning of novel procedures. In domains such as multidigit subtraction, procedural knowledge often begins to develop first because of children's repeated exposure to procedures before domain concepts (e.g., Hiebert & Wearne, 1996). Furthermore, although one type of knowledge may begin to emerge first, this knowledge facilitates acquisition of the other type of knowledge, thus leading to positive correlations between the two. Thus, statements about which type of knowledge develops first must be qualified because of the partial nature of the knowledge, the impact of previous experience on the sequence of acquisition, and the mutually supportive relations between the two types of knowledge.

Mechanisms Underlying Relations Between Conceptual and Procedural Knowledge

What mechanisms are responsible for the knowledge change described by the iterative model? This study provides evidence for one important mechanism and offers clues to others. First we consider potential mechanisms linking conceptual knowledge to gains in procedural knowledge, and then we consider potential mechanisms linking procedural knowledge to gains in conceptual knowledge.

Conceptual Knowledge → Gains in Procedural Knowledge

Forming correct problem representations is one mechanism underlying the influence of conceptual knowledge on improvements in procedural knowledge. First, children who had greater conceptual knowledge at pretest subsequently represented more problems correctly. Forming correct problem representations, in turn, was related to improvements in procedural knowledge. In Experiment 1, frequency of correct problem representations was a mediator of the relation between prior conceptual knowledge and improved procedural knowledge. In Experiment 2, representational supports, along with feedback on number line problems, enabled children to generate correct procedures for solving the intervention problems and to sustain these gains in procedural knowledge at posttest. Thus, given relevant problem-solving experience and feedback, conceptual knowledge and representational support enhance problem representation; improved problem representation, in turn, leads to changes in procedural knowledge.

Improved problem representation is not the only mechanism underlying the link from initial conceptual knowledge to gains in procedural knowledge. In Experiment 1, correct problem representation was only a partial mediator of the relation between initial conceptual knowledge and improvements in procedural knowledge, and alternative mediators were not considered. In Experiment 2, although children were given direct support for forming correct problem representations, initial conceptual knowledge continued to influence gains in procedural knowledge. These findings raise the question: Through what mechanisms other than improved representation might initial conceptual knowledge influence improvements in procedural knowledge?

One such mechanism may be improved choices among competing procedures. Conceptual knowledge can guide people's choices among alternative procedures (see Crowley, Shrager, & Siegler, 1997; Shrager & Siegler, 1998). Increasing use of correct procedures (and decreasing use of incorrect procedures) in turn, is a crucial component of improved procedural knowledge (Lemaire & Siegler, 1995; Rittle-Johnson & Siegler, 1999). In this study, children's conceptual knowledge about decimal fractions may have guided them to choose correct procedures more often and incorrect procedures less often.

Another potential mechanism underlying the relation between conceptual knowledge and gains in procedural knowledge is that conceptual knowledge guides adaptation of existing procedures to the demands of novel problems. In both experiments, children's prior conceptual knowledge was more strongly related to transfer of procedures to novel problems than to use of procedures on the type of problems on which children received feedback (i.e., the problems used in the intervention phase and the posttest). Children may use their conceptual knowledge to evaluate the relevance of known procedures to novel problems and to transform the known procedure for use on the new problems (e.g., Anderson, 1993).

Procedural Knowledge → Improved Conceptual Knowledge

Improvements in problem representation may also be one way in which gains in procedural knowledge lead to improved conceptual knowledge. For example, a procedure based on locating the tenths digit before considering the other digits requires use of a composite representation of decimal values. Forming such representations could lead to greater understanding of the concept of place value, because the digit's place in the number is a crucial component of the representation. The relations among conceptual knowledge, problem representation, and procedural knowledge may all be bidirectional.

At least four other mechanisms may contribute to the relation between improved procedural knowledge and improved conceptual knowledge. First, using conceptual knowledge to generate a procedure may strengthen the conceptual knowledge and facilitate its future retrieval. Within activation-based theories of cognition, using knowledge increases its activation and facilitates recall (Anderson, 1993). In this study, children began with a partial understanding of several decimal fraction concepts. After working on the intervention problems, children made improvements in their understanding across a range of concepts. These diverse gains suggest that children's knowledge of each concept was strengthened.

Second, gains in procedural knowledge may make attentional resources available for children to devote to other processes (Geary, 1995; Silver, 1987). As children use fewer mental resources to solve the immediate problem, they should have more resources available for planning, observing relations between problems, generating new procedures, and reflecting on the problems and the concepts underlying them. A recent model of strategy choice embodies the view that increased knowledge of procedures leads to more attentional resources being devoted to such higher level processes (Shrager & Siegler, 1998). The freed mental resources also may lead to increased conceptual understanding of the task.

Third, improvements in procedural knowledge may highlight children's misconceptions. For example, children often misapply their understanding of whole numbers to decimal fractions (Hiebert, 1992; Resnick et al., 1989). Using a correct procedure on number line problems, and observing the outcomes, may help children recognize some of the misconceptions that supported their previous, incorrect procedures. Indeed, in both experiments, children were less likely to treat decimal fractions as whole numbers after the intervention. For example, children used the whole number rule less often on the magnitude comparison task on the posttest than on the pretest. Thus, procedural knowledge may influence gains in conceptual knowledge by helping children to identify and eliminate misconceptions.

Finally, reflection on why procedures work may also link gains in procedural knowledge to gains in conceptual knowledge. Students who try to explain the conceptual basis of facts and procedures that they encounter learn more than those who do not. Prompting children to generate such explanations can lead to improved learning (Chi et al., 1989; Chi, De Leeuw, Chiu, & LaVancher, 1994; Pine & Messer, 2000; Renkl, 1997; Siegler, 1995). All the children in this study were encouraged to explain the correct solutions during the instructional intervention. Generating these explanations may have helped children understand the concepts underlying the procedures they were using.

Overall, there are multiple potential mechanisms underlying the bidirectional relations between conceptual and procedural knowledge. Conceptual knowledge may influence gains in procedural knowledge by improving problem representation, increasing selection of correct procedures, and facilitating adaptation of known procedures to the demands of novel problems. Gains in procedural knowledge may produce gains in conceptual knowledge through improved problem representation, strengthening of conceptual knowledge, increased availability of mental resources, identification of misconceptions, and reflection on why procedures work. Future research is needed to assess the viability of each of these potential mechanisms.

Educational Implications

The present findings have at least three important implications for education. First, competence in a domain requires knowledge of both concepts and procedures. Developing children's procedural knowledge in a domain is an important avenue for improving children's conceptual knowledge in the domain, just as developing conceptual knowledge is essential for generation and selection of appropriate procedures. Current reforms in education focus on teaching children mathematical concepts and often downplay the importance of procedural knowledge (e.g., NCTM, 1989). Furthermore, some educators treat the relations between conceptual and procedural knowledge as unidirectional (e.g., Putnam et al., 1992). They claim that conceptual knowledge can support improved procedural knowledge but suggest that the reverse is not true. In contrast, we found that the relations between conceptual and procedural knowledge are bidirectional and that improved procedural knowledge can lead to improved conceptual knowledge, as well as the reverse. Thus, it is important that both types of knowledge are inculcated in the classroom.

A second educational implication of the present findings is that identifying the processes used by good learners is a powerful resource for designing educational interventions. In Experiment 1, children who made large learning gains used composite, not common unit, representations of decimal fractions. This nonintuitive finding guided the design of the prompts manipulation in Experiment 2, which proved to be an effective tool for improving learning. In particular, helping children to think of decimal fractions as having a certain number of tenths, a certain number of hundredths, and so on, led to improvements in children's ability to solve decimal fractions problems. In contrast, in Experiment 1, the common unit representation, which children are taught in most classrooms (e.g., 0.45 is read as 45 hundredths), was unrelated to success at problem solving. Perhaps teachers should help children develop a composite representation of decimal values by focusing children's attention on the tenths digit and providing external representations of the meaning of tenths. Teaching of the meaning of hundredths and thousandths within multidigit decimals can proceed from there. As this example illustrates, identifying the learning processes of good learners, and supporting these processes in students who use weaker methods, can enhance children's learning.

A third instructional implication is that supporting correct representation of problems is an effective tool for improving problem-solving knowledge. In Experiment 2, children who received representational support made greater gains in procedural knowledge than children who did not. However, representational supports must be designed carefully. Children needed to be encouraged to apply the composite representation to both the decimal fraction and the number line for optimal learning and transfer to occur.

Conclusion

Children's conceptual and procedural knowledge develop iteratively. Rather than development of one type of knowledge strictly preceding development of the other, conceptual and procedural knowledge appear to develop in a hand-over-hand process. Gains in one type of knowledge support increases in the other type, which in turn support increases in the first. One key mechanism underlying these relations is change in problem representation. In the present study amount of improvement in problem representation varied as a function of initial individual differences in conceptual knowledge, and amount of improvement in problem representation predicted individual differences in acquiring procedural knowledge. Furthermore, supporting correct problem representation led to greater gains in procedural knowledge. To understand how knowledge change occurs one must consider the interrelations among conceptual understanding, procedural skill, and problem representation. Carefully analyzing these relations, and using the analysis to inform instruction, can help children learn.

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