Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction is Important to Teachers

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Abstract: The terms conceptual knowledge and procedural knowledge are often used by teachers and never more so than when discussing how teachers teach, and children learn mathematics. This paper will look at literature regarding conceptual and procedural knowledge and their place in the classroom, to offer teachers and teacher educators' advice on some of the more pressing issues and understandings around them. A thorough synthesis of extant and seminal literature will provide advice to teachers and teacher educators on how a deeper insight into conceptual and procedural knowledge could improve the quality of mathematics teaching.

Keywords Conceptual knowledge, Procedural knowledge, Teacher understanding

Introduction

Recently, when working with a group of highly motivated, mathematically perceptive, experienced primary and lower secondary teachers, questions were raised around conceptual and procedural knowledge. The teachers were asked if it was always better to teach to develop conceptual knowledge or procedural knowledge, and then offered the opportunity to justify and qualify their responses. This question was raised in the knowledge that it was a 'loaded' question, one that might provoke responses which were either strong, or guarded, due to expected conventions about teaching and learning. What emerged after some discussion, was that the collective understanding regarding conceptual and procedural knowledge, seemed worthy of further scrutiny. In essence, the teachers indicated that they carried the belief that conceptual and procedural knowledge were mutually exclusive, and that conceptual knowledge was, without exception, more appropriate or necessary than procedural knowledge.

What do we mean by Concepts and Conceptual Knowledge?

According to Westwood (2008) a concept can be defined as "a mental representation that embodies all the essential features of an object, a situation, or an idea. Concepts enable us to classify phenomena as belonging, or not belonging, together in certain categories" (p. 24). Chinn (2012) defined concepts as characteristics that determine either the inclusion or the exclusion of something from a set or class. The focus is on classifying, categorising, ordering and on labelling. Concepts, according to Rittle-Johnson and Koedinger (2009), are ideas that are generalised from specific instances and that govern a domain; for example, place-value. If a student can recite the place value of a number as an isolated piece of information due to remembering the 'verbal labels' of each position, this is not conceptual knowledge. It becomes conceptual once that knowledge is linked to other knowledge, such as the grouping of objects by ten and the multiplicative nature of each of the places.

Bruner (1966) determined that concepts are developed through a series of stages. It commences with the 'enactive' stage where learning involves concrete experiences. Secondly is the 'iconic' stage. The 'iconic' stage is the where pictorial and other graphic representations are engaged. The 'symbolic' stage is the final stage and where abstract notation, and symbols are considered apposite for carrying meaning to the learner. This progression was further developed by Biggs and Collis (1982; 1991) when proposing their SOLO Taxonomy with Multimodal Functioning. The seminal work of Bruner (1966) and Biggs and Collis (1982; 1991) is still extant and underpins the contemporary instructional practice of CRA (Concrete-Representational-Abstract). CRA was originally visualised as a way to work with students with Learning Difficulties by employing graduated instruction (Strickland & Maccini, 2013). However, CRA, which in the literature is also referred to as CPA (Concrete-Pictorial-Abstract), proved to be an effective strategy for mainstream students to gain an understanding of needed mathematical concepts and skills (Agrawal & Morin, 2016; Flores, 2010; Miller & Kaffar, 2011).

Conceptual knowledge (notably characterised by Skemp, 1978, as Relational Knowledge) may be visualised as a connecting web of relationships (Miller & Hudson, 2007; Rittle-Johnson & Schneider, 2015). This connection can be between two previously learned mathematical ideas or concepts, or be a connection between a concept previously learned and a concept newly learned; "the principles which govern a domain" (Rittle-Johnson, Fyfe, & Loehr, 2016, p. 576). Some researchers (e.g. Hiebert, 1986, Rittle-Johnson & Schneider, 2015) have characterised it as being knowledge, where the rich links and relationships are as equally vital as the separate bits of information they join. However, Baroody, Feil, and Johnson (2007) asserted that when defining conceptual knowledge as being knowledge about facts, principles and generalisations, there is no necessity for the knowledge to be richly related. Rather, the research of Baroody, Feil, and Johnson (2007) and others (e.g. diSessa, Gillespie, & Easterly, 2004; Schneider & Stern, 2009) advocates that the conceptual knowledge of novices can often be disjointed, and can require time to become integrated, and that the richness of the connections increases with developing expertise. Scrutiny of Baroody, Feil, and Johnson's (2007) claim may lead to proposing a position with regards to the type of knowledge, conceptual or procedural, and also of the qualities of each type.

Richland, Stigler and Holyoak (2012) characterised conceptual knowledge as the attainment of expert facility of the conceptual structure of a domain. The use of the word structure is informed through the work of Bruner (1966), who wrote about the role of structure in thinking and learning in the development of concepts. Bruner identified four functions that concepts perform in helping us organise people's perceptions and understanding. Concepts:

- provide structure for a discipline
- provide a framework within which details can be more readily understood and remembered
- are the primary bridges which make transfer of learning possible; and
- provide the framework for lifelong learning.

Researchers (Mason, Stephens, & Watson, 2009; Mulligan & Mitchelmore, 2009) have written about mathematical structure often being expressed in the form of a generalisation or a relationship, which is seen to be constantly true in a domain. Their deliberate use of the word relationship and the use of this word by other researchers (e.g. Hiebert, 1986; Star, 2005), offered a connection to the definitions given for conceptual knowledge. Clark (2011) saw concepts as the most powerful and useful cognitive tools available to people, as concepts have the 'capacity' of organisation and association. In essence a concept is an idea that is well enough understood to allow other ideas to be connected with it and become part of a web of understanding. Such connections and webs often lead to the formation of conceptual knowledge.

What do we mean by Procedures and Procedural Knowledge?

Procedures are a series of steps and/or actions employed to achieve a task or reach a goal (Hiebert & Lefevre, 1986; Rittle-Johnson, 2017; Rittle-Johnson, Schneider, & Star, 2015). Adopting this definition, without taking regard of the qualities of procedural knowledge (Table 2), could lead to what Skemp (1978) referred to as learning "rules without reason" (p. 9). Martin (2009) warned that executing procedures in such a mechanical fashion which employs rules without reason can often lead to peculiar and unreasonable solutions. Written algorithms (for example dividing a 4-digit number by a 2-digit number) are an often employed procedure, as are actions which have been suitably arranged to solve a problem, for example equation-solving steps (Rittle-Johnson & Schneider, 2015). In essence a procedure is a routine, but it can be either thoughtfully considered, or executed with little consideration.

Procedural knowledge is characterised by some researchers (Canobi, 2009; Miller & Hudson, 2007; Rittle-Johnson & Schneider, 2015) as the capacity to follow steps in sequence to solve mathematical problems or reach a mathematical goal. This can comprise a familairity with, and a knowledge of, the system of symbols to construct algorithms, but can also pertain to a knowledge of procedural rules necessary to solve problems (Hiebert, & LeFevre, 1986; Rittle-Johnson & Schneider, 2015). Baroody, Feil, and Johnson (2007) observed that procedures can often be interconnected or embeded within other procedures, and disagree with teachers who may view procedural knowledge to be devoid of relationships. Again, it appears prudent to reflect on the qualities of procedural knowledge, rather than to just accept a shallow, illconsidered, and perhaps sometimes unconsidered characterisation of this type of knowledge.

Conceptual and Procedural Knowledge in Creating Mathematically Powerful Classrooms

Improving the quality of mathematics learning and teaching is a pressing matter across the globe (Cobb & Jackson, 2011), and yet, how to support instructional improvement is an area which is not researched particularly well (Cobb & Jackson, 2011; Coburn, Russell, Kaufman, & Stein, 2012; Cohen, Moffitt, & Goldin, 2007; Stein, 2004) and needs to be considered an important topic for researchers (Cobb & Jackson, 2011). One field of research where there is substantial work, regards teacher impact on the academic success of students (Charalambous, Hill, & Mitchell, 2012; Chetty, Friedman, & Rockoff, 2014). How student success can be encouraged has engendered further research into effective teaching practice and much of this research has focussed on the attributes or characteristics of effective teachers of mathematics, including factors such as: subject matter knowledge (Ball, Thames, & Phelps, 2008; Cobb and Jackson 2011); pedagogical content knowledge (Shulman 1986); teacher efficacy (Young-Loveridge & Mills 2009; Zambo & Zambo 2008); and teacher confidence, attitudes and beliefs (Swars, Hart, Smith, Smith, & Tolar, 2007). There has also been substantial research into how students learn mathematics (Daro, Mosher, & Corcoran, 2011). There is arguably no greater issue in the teaching and learning of mathematics more pressing than the decisions teachers make about whether to teach procedurally or conceptually.

The connections between how students learn mathematics and the success they might encounter through successful teaching practices is one that asks teachers and other mathematics educators to consider how learning is best facilitated. Such questions are complex, as the practice of teaching is complex (Clarke & Pittaway, 2014; Danielson, 2013, Hattie, 2015). The practice of teaching asks teachers to make choices with regards to instructional strategies on a daily basis, choices which need to be informed. These strategies should always be focussed on developing mathematically powerful classrooms (Hattie, 2015; Schoenfeld, 2014). Whether considering Schoenfeld's five dimensions of powerful classrooms, or Hattie's distinction between the 'expert' teacher and the 'experienced' teachers, or other researchers who articulate indicators of teachers who provide quality instruction (Charalambous, Hill & Mitchell, 2012; Hill, Ball & Schilling, 2008; Hill, Rowan & Ball, 2005), the need for focussed, quality instruction is central. Focussed, quality instruction requires teachers to make judgements, and one key judgement is the consideration of the place of procedural knowledge and conceptual knowledge in the teaching of mathematics.

That many students were not developing a conceptual understanding of mathematics, and that consequently this was critically inhibiting their capacity to transfer and generalise mathematics was a concern of Richland, Stigler and Holyoak (2012). They posited that this paucity in the development of conceptual understanding was resulting in students who despite having success with mathematics in high school, found a subsequent need to receive remedial assistance while attending community colleges. Further research (Givvin, Stigler, & Thompson, 2011; Stigler, Givvin, & Thompson, 2010) concluded that the mathematical knowledge of these students was largely procedural and left the students with ineffectual mathematical reasoning and a want to conduct incorrect or partially correct procedures. Such a reliance on procedural knowledge was a further issue, in that many of these students used their two-year associate's degree (a degree which is an alternative pathway into tertiary study) gained at community college, as a springboard to four-year degrees, which often required a knowledge of mathematics that was more conceptual (The Princeton Review, 2017).

If it is the case that the school system is producing students who are procedural in their approach to solving mathematical problems and who employ ineffective reasoning, is there an approach to teaching and learning to ameliorate or even remedy this? Such a question only has potency if it is acknowledged that a concentration on developing procedural knowledge is not the sole purpose of mathematics education, but also recognises the importance of conceptual knowledge. Research is unambiguous in accepting that both conceptual knowledge and procedural knowledge are important (Hiebert & Grouws, 2007; Rittle-Johnson, Schneider & Star, 2015). This acceptance that both forms of knowledge are important allows the debate to move to determining the relationship between the two. Although this debate still receives contemporary attention (Alcock et al., 2014; Rittle-Johnson, 2017; Rittle-Johnson & Koedinger, 2009; Schneider, Rittle-Johnson & Star, 2011) it is one that has some history (e.g. Resnick & Ford, 1981; Sowder, 1998). In the past, the terms have been couched differently (Table 1), but regardless of the labels, the divisions regarding the types of knowledge are consistent. (Hiebert & Lefevre, 1986).

		Procedural	Conceptual
Skemp	1976	Instrumental	Relational
Piaget	1978	Successful action	Conceptual understanding
Gelman & Gallistel	1978	Skills	Principles
Resnick	1982	Syntax	Semantic
Tulving	1983	Episodic memory	Semantic memory
Anderson	1983	Procedural	Declarative
Van Lehn	1983	Teleologic	Schematic
Baroody	1984	Mechanical	Meaningful

Table 1: Differing Terms for Procedural and Conceptual Knowledge - Hiebert & Lefevre (1986, p. 14)

Is Conceptual Knowledge always 'deeper' than Procedural Knowledge?

Star (2005) proposed a matrix to represent the types and qualities of conceptual and procedural knowledge (Table 2). The matrix indicates that for each knowledge type (conceptual and procedural) there is the possibility of developing a superficial knowledge or a deep knowledge. However, Star noted that with the prevailing, yet often erroneous, interpretation of what conceptual knowledge (seen by some as solely deep connected learning) and procedural knowledge (seen by some as solely step by step, prescriptive learning) evokes, it is challenging to determine an illustration or articulation of deep-procedural, or shallow-conceptual knowledge. Due to the range of qualities contained within conceptual and procedural knowledge, Kieran (2013) declared the dichotomy between them to be fundamentally unsound. Kieran (2013) writes that "…during any period of elaboration, procedures are conceptual in nature" (p. 212) and that procedures are regularly being extended and revised, and therefore updated, by means of conceptual elements.

	Knowledge quality	
Knowledge type	Superficial	Deep
Procedural	Common usage of procedural knowledge	?
Contextual	?	Common usage of conceptual knowledge

Table 2: Types and Qualities of Conceptual and Procedural Knowledge - Star (2005)

As Star's (2005) matrix did not attempt to illustrate the connection between these two types of knowledge, Baroody, Feil, and Johnson (2007) proposed a reconceptualisation to represent the different types and qualities of conceptual and procedural knowledge which recognised the connections. This reconceptualisation of Star's (2005) model included constructs which they called, routine expertise, and adaptive expertise. Routine expertise is where there is a superficial conceptual and/or procedural knowledge, which is able to be applied to familiar situations, but not unfamiliar ones, or to new tasks. Adaptive expertise is where both conceptual and procedural knowledge is deep, and where that knowledge can be aplied creatively, flexibly and appropriately to all situations, familiar or new.

Baroody, Feil, and Johnson (2007) used their reconceptualisation of Star's (2005) perspective to inform their own representation of the dependency between procedural and conceptual knowledge (Figure 1). This further reconceptualisation of Star's model (2005) was deemed necessary due to unease that Star equated deep-knowledge only with richly connected knowledge, but lacked other aspects of knowledge quality (level of structure/ degree of organisation; abstractness; and accuracy) and knowledge completeness (connections to everyday situations and applications).

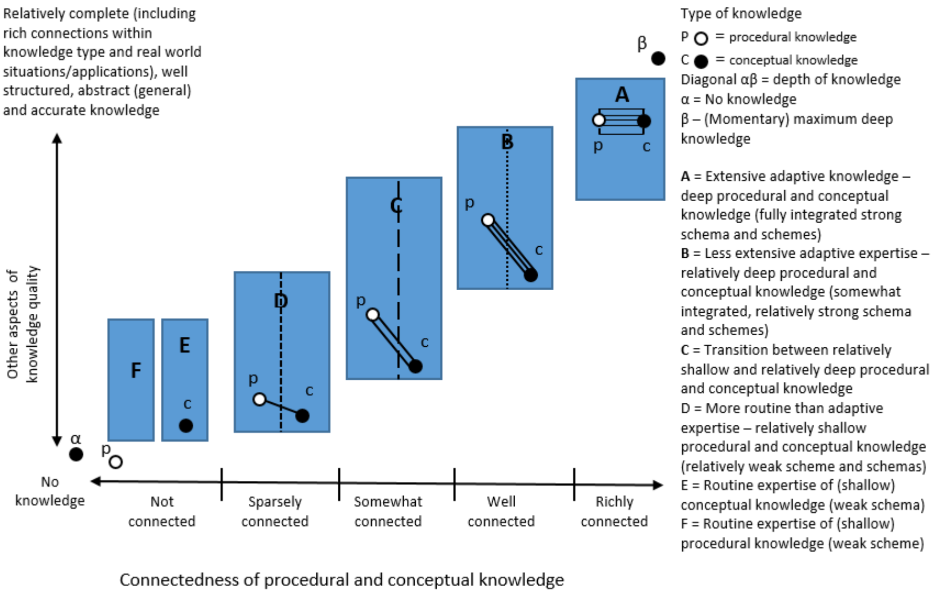

Figure 1. The mutually dependent relationship between procedural and conceptual knowledge suggested by a model of adaptive reasoning - Baroody, Feil, & Johnson (2007, p. 124)

Which comes first, Procedural or Conceptual Knowledge?

The unstated belief which predicated Baroody, Feil, and Johnson's (2007) model (Figure 1) is that students need knowledge of both concepts and procedures and that they have an influence on each other. It is the notion of this relationship that presents further issues. Rittle-Johnson and Schneider (2015) offered four differing views as to the relationship between procedural and conceptual knowledge. These four are the;

- procedure-first view (a uni-directional view)
- the concepts-first view (a second uni-directional view)
- the inactivation view (where both conceptual and procedural knowledge is thought to develop independently of each other), and
- the iterative view where the causal relationship is seen as being bi-directional, that is, increases in one, generates increases in the other.

This iterative, bi-directional view is now considered to be the most accepted (Rittle-Johnson & Schneider, 2015) with research finding correlations between procedural and conceptual knowledge across a range of domains and ages (Cowan et al., 2011; Dowker, 2008; Durkin, Rittle-Johson, & Star, 2011; Hallet, Nunes, & Bryant, 2010; Hecht & Vagi, 2010; Patel & Canobi, 2010; Star & Rittle-Johnson, 2009). In synthesising the available research, Rittle-Johnson and Schneider (2015) concluded that although the relationship between conceptual and procedural knowledge is bi-directional, it is not always symmetrical, and that, at times, conceptual knowledge is stronger and more consistent in supporting procedural knowledge, than the reverse.

As the reseach indicates that the two types of learning are iterative, a question of an optimal sequence is raised, that is, whether procedural knowledge or conceptual knowledge should be introduced first, or if indeed it matters. Although Grouws and Cebulla (2000) and the National Council of Teachers of Mathematics (NCTM, 2014) overtly support the concept-first iterative approach, most researchers appear to be a little more circumspect with their support. Although in promoting the iterative process, researchers (e.g. Canobi, 2009; Khashan, 2014; Rittle-Johnson & Koedinger, 2009; Rittle-Johnson & Schneider, 2015; Star, 2002) appeared reticent to deem that one type of knowledge should *always* precede the other, it seems that the examples they provided regarding instructional methods, usually start with conceptual knowledge before procedural knowledge.

So, should it always be a Concept-first Approach?

Circumspection regarding the concept-first approach is evident through the research of Rittle-Johnson and Koedinger (2009) and Rittle-Johnson, Schneider, and Star (2015) who state that it would be *beneficial* (my emphasis) for the early introduction of procedures to occur after an initial concept lesson, and that conceptual knowledge often supports procedural knowledge. Even using a guarded word such as beneficial, (rather than the suggestion that it is advisable, or even desireable) this displays a disposition towards conceptual knowledge predicating procedural knowledge.

With little empirical evidence to support a concept- first iterative approach, why then has it enjoyed such a pervasive adoption? Firstly, the evidence there is, shows that in developing conceptual knowledge of mathematics, students who are taught for a conceptual understanding followed by a procedural understanding, outperform students who are instructed for procedural then coneptual knowledge (Pesek & Kirshner, 2000). Pesek and Kirshner's (2000) study also reinforced the research of Hiebert (1999), who asserted that once students have memorised and practised procedures (including written algorithms, that they do not necessarily understand) they have less motivation to comprehend their meaning or the reasoning behind them. This indicates that trying to create a situation or environment for bi-directional iteration to occur requires a concept-first approach.

A further argument as to the need for a conceptual first approach sits with the research which challenges a longstanding preoccupation with breaking mathematics knowledge into small pieces, and asserts that doing so, may be counterproductive to deep learning (Pellegrino & Hilton, 2012). Conley (2014) stressed that the approach should rather be about allowing the students the opportunity to grasp the 'big picture', that is, developing conceptual knowledge. Other research also posits that although 'experts' do know more 'facts', crucially it is that the facts are connected and organised into meaningful schemas or patterns, a characteristic of conceptual knowledge, that is important (Ericsson, Charness, Feltovich, & Hoffman, 2006). It is this schematised conceptual knowledge which allows them to select and remember relevant information and extract levels of meaning not apparent to novices (Chi, Glaser, & Rees, 1982). Further, this organisation allows for greater transfer; what was learned, can be transferred into new situations more quickly (Schwartz, Bransford, & Sears, 2005).

Consideration also needs to be given, regarding issues which impact on the knowledge types and therefore on the efficacy of the learning which can be expected. Pesek and Kirshner (2000) pursued the issue of interference of prior learning, writing that there are two types of interference cognitive and attitudinal (although they mention a third, metacognitive interference, which is an intermediate between their two stated types). Cognitive interference, considered by Pesek and Kirshner as being the more important interference, is when previous understandings of something are so poweful they obtrude into subsequent learning. An example of this could be when a student has generalised the procedural undertanding of the equals sign (=) standing for "give me an answer" (Knuth, Stephens, McNeil, & Alibali, 2006). This understanding is quite appropriate when working in arithmetic, but is highly problematic later in school with the requirement to solve equations (Booth, Barbieri, Francie Eyer, & Paré-Blagoev, 2014), as the idea of balance across two sides of an equation is a foundational concept of equivalence (Chesney & McNeil, 2014). Therefore if the 'arithmetic' understanding of the equals sign is persistent, then future teaching and learning is obfuscated. This is a case where procedural knowledge might interfere with a conceptually broader, more complex, and arguably mathematically more important, understanding.

Attitudinal interference is where a student's previously acquired opinions and attitudes block comprehensive engagement with a topic and therefore impede potential for learning (Pesek & Kirshner, 2000). For example if the student has been exposed to pedagogical practices which they have found unengaging. Many teachers who teach from a predominantly procedural knowledge standpoint, would likely employ textbooks as a significant source of their teaching. According to Boaler (1998) this approach emphasises computation and procedures and encourages limited, school-bound and inflexible learning, not a description which sits in concordance with conceptual knowledge. It may then be reasonable to assert that favouring procedural knowledge through the use of textbooks, and by extension, didactic teaching may promote attitudinal interference to learning, particularly when conceptual knowledge is desired. What has also been observed in detailed analyses of mathematics textbooks is that they tend to adopt a dominant mathematical teaching method which revolves around presenting archetypal tasks and then suggesting methods of solution. These solutions most commonly are in the form of algorithmic templates such as rules, methods and solved examples tasks. Such methods of solution produce learning in the short-term but there are data indicating that they fail to enhance students' long-term conceptual knowledge (Wirebring, et al., 2015).

In combining the research that promotes a concept-first iterative approach to learning, with an understanding of the possible interferences of prior learning, a sound basis is built for recommending that although learning should be iterative between conceptual and procedural knowledge, a concept-first iterative process is endorsed. This position being clearly foregrounded, and in order that the iterative process be properly explored, it should be acknowledged that not only does conceptual knowledge support the development of procedural knowledge, the research further indicates, that development of procedural knowledge supports the development of conceptual knowledge (Hiebert & Wearne, 1996; Peled & Segalis, 2005; Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Schneider, & Star, 2015). There is also evidence that gaining procedural knowledge can lessen the cognitive demands on working memory. This lessening of demand can then free the working memory to focus on conceptual knowledge development (Sweller, van Merrienboer, & Paas, 1998). These arguments accepted it should be noted that the extent to which gains in procedural knowledge support gains in conceptual knowledge, is markedly influenced by the nature of the practice or procedural instruction. Research (Canobi, 2009; McNeil et al., 2012) showed how sequencing arithmetic problems to encourage the identification of conceptual relationships proved to be efficacious, whereas when the problems were in random order this effect was not noted. The inference here is that thoughtful, deep-knowledge of procedure was required on the teachers' behalf, to support the conceptual knowledge (Rittle-Johson & Schneider, 2015). This deep-knowledge of procedures could actually be considered to show an understanding of connections and relationships, and therefore has qualities which are analogous to conceptual knowledge.

This acknowledgment of procedural knowledge should in no way be taken as support for a procedure-first, or procedure-only argument. Justifying an over-focus on procedures by asserting that it is a shortcut to computational fluency is a flawed argument. Researchers (Ginsburg, 1977; Skemp, 1978) have long declaimed that 'meaningful' memorisation is more effective than rote memorisation, and researchers such as Baroody, Feil, and Johnson (2007) claimed that linking procedural knowledge to existing conceptual knowledge is required. Linking procedural knowledge to conceptual knowledge can; make learning facts and procedures easier, provide computational shortcuts, ensure fewer computational errors, and promote efficiency. Procedural knowledge connected exclusively or largely with other non-conceptual knowledge tends to yield more error-prone, rigid, short-term, or isolated gains than would more conceptually connected procedural knowledge (Baroody, 2003; Carpenter, 1986). As Fuson, Kalchman, and Bransford (2005) wrote, when procedures are connected with underlying concepts, students are able to better retreive and employ them.

Conclusion and Discussion

Improving the quality of mathematics teaching and learning is a pressing issue across the globe (Cobb & Jackson, 2011). One area where there is significant research into improving the quality of mathematics teaching and learning is in regard to the impact that teachers have on student academic success (Charalambous et al. 2012; Chetty, Friedman, & Rockoff, 2014). How such academic success is accomplished has engendered further research into effective teaching practice (Daro, Mosher, & Corcoran, 2011). When examining teaching practice, one issue which arises is whether teachers should be teaching mathematics for procedural knowledge, conceptual knowledge, or a combination of the two (Rittle-Johnson & Schneider, 2015).

With due consideration of contemporary literature and reseach regarding procedural and conceptual knowledge, of what then teachers should be aware?

- We should be considering our practices to include Procedural knowledge *and* Conceptual knowledge not Procedural knowledge *or* Conceptual knowledge.
- Procedural knowledge and conceptual knowledge are both important and help to strengthen each other.
- Conceptual knowledge in most cases should precede procedural knowledge.
- There is more chance of students developing a conceptual knowledge if they start with conceptual knowledge and then move to procedural knowledge. Moving in the opposite direction, procedural to conceptual, has the risk that students will not work towards conceptual knowledge.
- Both procedural and conceptual knowledge are more nuanced structures than many teachers realise. Both can be ‘shallow’ (superficial) or ‘deep’.
- Cognitive and attitudinal interference both need to be recognised and acknowledged. Conceptual knowledge appears more suited in avoiding the two sources of interference for learning.
- Meaningful memorisation is valuable. Memorisation is important (for one thing it frees up the working memory) but memorisation *with* meaning is far more efficacious.

Teachers deserve to be made aware of the contemporary research and literature with regards to procedural and conceptual knowledge, in order that they might make informed decisions when creating their teaching and learning environment. Anecdotally there is a good deal of misinformation regarding the efficacy of both knowledge forms amongst teachers, and this is unhelpful. For example, the false positioning of procedural and conceptual knowledge as being two extremes of a continuum, rather than them having the capacity to occupy different places on that continuum depending on the situation, does little to acknowledge their complexity. Conversations around the teaching and learning of mathematics need to be framed on sound understandings of procedural and conceptual knowledge, rather than on limited, and sometimes erroneous, (mis)understandings. In answer to the title of this article it should be Conceptual knowledge *and* Procedural knowledge, the when and where, that is determined by the professional call of the teacher.

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